SMALL CANCELATION: EXERCISE SHEET 2

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(1) Let X be a 2-complex. Prove that if X satisfies C(p), (respectively C'($\frac{1}{p}$)), then every reduced disk diagram of X satisfies C(p), (respectively C'($\frac{1}{p}$)). Does this also hold for non-reduced disk diagrams?

(2) Prove that if every reduced disk diagram satisfies C(p), (respectively C'($\frac{1}{p}$)) then X satisfies C(p), (respectively C'($\frac{1}{p}$)).

(3) Show that the 2-complex obtained from a hexagon by identifying opposite edges together in an orientation preserving manner satisfies C(6) and T(3).

(4) Let X be $\mathbb{S}^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$. A regular tiling of X is a 2-complex structure on X where the 2-cells are regular polygons, all of the same isometry type, and the 1-cells and 0-cells correspond to the edges and vertices of 2-cells. For example, there are five such tilings of $\mathbb{S}^2$ up to isometry and they are correspond to the Platonic solids. Up to similarity, there are three such tilings of $\mathbb{R}^2$: one by equilateral triangles, one by squares and one by hexagons. There are infinitely many regular tilings of $\mathbb{H}^2$ up to isometry.

Note that a regular tiling by p-gons in which each vertex has degree q satisfies C(p) and T(q). What do you notice about the values of $\frac{1}{p} + \frac{1}{q}$ for tilings of $\mathbb{S}^2$, $\mathbb{R}^2$ and $\mathbb{H}^2$? Why are there finitely many regular tilings of $\mathbb{S}^2$ and $\mathbb{R}^2$ and yet infinitely many of $\mathbb{H}^2$?

(5) Prove that if a 2-complex X satisfies T(5) then X has no pieces of length greater than 1.

(6) Let $\hat{X} \to X$ be a covering map. Prove that $\hat{X}$ satisfies C(p), (respectively C'($\frac{1}{p}$) or T(q)) if and only if X satisfies C(p), (respectively C'($\frac{1}{p}$) or T(q)).

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