MATH 4530: HOMEWORK #6

- Due Friday, October 13th at 11pm.
- To be submitted on the course gradescope.
  - Please remember to indicate which pages your solutions to each problem appear on.
- §n: #m refers to exercise number m from the exercise list following Section n of the textbook (Topology, Second Edition by James R. Munkres).
- You are encouraged to discuss the problems in groups. *However, you must write your solutions individually!*

Problems.

(1) §23: #2
(2) §23: #6
(3) §23: #11
(4) Let $\{X_\alpha\}_\alpha$ be a family of topological spaces. Recall that the disjoint union $X = \bigsqcup_\alpha X_\alpha$ has underlying set $\bigcup_\alpha (\{\alpha\} \times X_\alpha)$ and has topology

$$\mathcal{T} = \big\{U \subset X \mid U \cap \big(\{\alpha\} \times X_\alpha\big) \text{ is an open subset of } \{\alpha\} \times X_\alpha, \text{ for each } \alpha\big\}$$

where $V \subset \{\alpha\} \times X_\alpha$ is open if and only if $V = \{\alpha\} \times V'$ for some open subset V' of X_α.
  (a) Prove that the disjoint union topology is the finest topology on $\bigsqcup_\alpha X_\alpha$ for which each inclusion map $\{\alpha\} \times X_\alpha \hookrightarrow X$ is continuous.
  (b) Prove that if each X_α is a one-point space then the disjoint union topology on $\bigsqcup_\alpha X_\alpha$ is discrete.
  (c) Prove that the disjoint union of at least two non-empty spaces is disconnected.

(5) Sometimes we would like to define an equivalence relation on a set X without having to explicity define every pair of related elements. The *equivalence relation generated by* a set of unordered pairs $\mathcal{P} = \big\{\{x_\alpha, y_\alpha\}\big\}_\alpha$ of elements of X is the equivalence relation $\sim$ given by $x \sim y$ if and only if there exists $x_1, x_2, x_3, \ldots, x_n \in X$ with $n \geq 1$ such that $x = x_1$, $y = x_n$ and $\{x_i, x_{i+1}\} \in \mathcal{P}$ for each $i \in \{1, 2, 3, \ldots, n-1\}$.
  (a) Prove that $\sim$ is indeed an equivalence relation.
  (b) Prove that $\sim$ is the smallest equivalene relation for which $x_\alpha \sim y_\alpha$ for every α. (This amounts to proving that any equivalence relation $\approx$ for which $x_\alpha \approx y_\alpha$ for every α also satisfies: for every $x, y \in X$, if $x \sim y$ then $x \approx y$.)

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(c) Let X be the disjoint union of a one point space $\{p\}$ and three copies of the interval $[0,1]$. For convenience, let r_i denote the copy of $r \in [0,1]$ coming from the ith copy of $[0,1]$. For example, 0_2 is the endpoint 0 of the second copy of $[0,1]$ in X. Let $\sim$ be the equivalence relation on X generated by $\big\{\{0_1, p\}, \{0_2, p\}, \{0_3, p\}\big\}$.
Find a subspace of $\mathbb{R}^2$ that is homeomorphic to the identification space $X/\sim$. Make an image of (e.g. draw) this subspace. Prove that your subspace is indeed homeomorphic to $X/\sim$.