MATH 4530: HOMEWORK #7

- Due Friday, October 20th at 11pm.
- To be submitted on the course gradescope.
  - Please remember to indicate which pages your solutions to each problem appear on.
- §n: #m refers to exercise number m from the exercise list following Section n of the textbook (Topology, Second Edition by James R. Munkres).
- You are encouraged to discuss the problems in groups. *However, you must write your solutions individually!*

Problems.

(1) §23: #10
(2) §24: #1
(3) §24: #8
(4) §24: #10
(5) §24: #11
(6) We will define the notion of an *n-dimensional cellular complex* by induction on $n \in \mathbb{N}$.

Base case $n = 0$: A *0-dimensional cellular complex* $X^{(0)}$ is a topological space whose topology is discrete.

Inductive step $n > 0$: Let $X^{(n-1)}$ be an $(n-1)$-dimensional cellular complex. Let $(f_\alpha \colon S^{n-1} \to X^{(n-1)})_{\alpha \in A}$ be a collection of continuous functions from the $(n-1)$-sphere

$$S^{n-1} = \left\{ (x_i)_{i=1}^n \in \mathbb{R}^n : \sqrt{\textstyle\sum_{i=1}^n x_i^2} = 1 \right\}$$

and recall that S^{n-1} is a subspace of the n-ball

$$B^n = \left\{ (x_i)_{i=1}^n \in \mathbb{R}^n : \sqrt{\textstyle\sum_{i=1}^n x_i^2} \leq 1 \right\}.$$

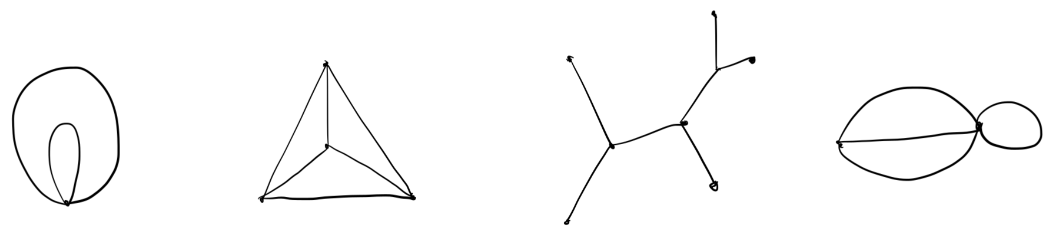
Consider the disjoint union $Y_n = X^{(n-1)} \sqcup \bigsqcup_{\alpha \in A} B^n$. For convenience, we write x_α, with $x \in B^n$, to denote the copy of x in the αth copy of B^n in the subspace $\bigsqcup_{\alpha \in A} B^n$ of Y_n. We also identify $X^{(n-1)}$ with its copy in Y_n. Let $\sim$ be the equivalence relation on Y_n generated by $\big\{\{x_\alpha, f_\alpha(x)\} : \alpha \in A \text{ and } x \in S^{n-1}\big\}$. Consider the identification space $X^{(n)} = Y_n/{\sim}$.

An *n-dimensional cellular complex* is a topological space $X^{(n)}$ obtained in this way from an $(n-1)$-dimensional cellular complex $X^{(n-1)}$ and a collection of continuous functions $(f_\alpha \colon S^{n-1} \to X^{(n-1)})_{\alpha \in A}$. The f_α are called the *attaching maps* of the n-cells of $X^{(n)}$. Since $\sim$ cannot relate distinct points of $X^{(n-1)}$, the spaces

 MATH 4530: HOMEWORK #7

$X^{(k)}$, $k \leq n$ are all naturally subspaces of $X^{(n)}$. The subspace $X^{(k)}$ is known as the *k-skeleton* of $X^{(n)}$. For the same reason, the interiors $B^k \setminus S^{k-1}$, $k \leq n$, of all the copies of balls that appeared in the construction of $X^{(n)}$ are all also naturally subspaces of $X^{(n)}$. Each of these and each point of $X^{(0)}$ is called a *cell* of $X^{(n)}$. In fact, the collection of cells of $X^{(n)}$ forms a partion of $X^{(n)}$.

Note that when $n = 1$, we have $S^{n-1} = S^0 = \{-1, 1\} \subset \mathbb{R}$ and $B^n = B^1 = [-1, 1] \subset \mathbb{R}$. So 1-dimensional cellular complexes are quotients of disjoint unions of 1-point spaces and closed bounded intervals of $\mathbb{R}$ obtained by identifying endpoints of intervals with points. Here are some examples of 1-dimensional cellular complexes, which are also known as *graphs*.



(a) Prove that graphs are Hausdorff. (In fact, this holds for all cellular complexes.)
(b) Describe the 2-sphere S^2 and the torus $S^1 \times S^1$ as 2-dimensional cellular complexes. As starting points, you may use the descriptions of the sphere and torus as identification spaces from class or from the textbook.