MATH 4530: HOMEWORK #8

- Due Friday, October 27th at 11pm.
- To be submitted on the course gradescope.
  - Please remember to indicate which pages your solutions to each problem appear on.
- §n: #m refers to exercise number m from the exercise list following Section n of the textbook (Topology, Second Edition by James R. Munkres).
- You are encouraged to discuss the problems in groups. *However, you must write your solutions individually!*

Problems.

(1) §25: #2
(2) §25: #5
(3) §26: #4
(4) §26: #5
(5) §26: #11
(6) For a finite set V, let $\mathbb{R}^V$ be the free real vector space on V. That is, the elements of $\mathbb{R}^V$ are formal sums $\sum_{v\in V} r_v v$ where each r_v is a real number. We have the standard operations of addition and scalar multiplication.

$$\sum_{v\in V} r_v v + \sum_{v\in V} r'_v v = \sum_{v\in V}(r_v + r'_v)v \qquad\qquad t\sum_{v\in V} r_v v = \sum_{v\in V}(tr_v)v$$

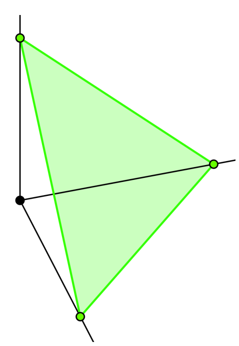

FIGURE 1. The simplex $\sigma_V \subset \mathbb{R}^V$ on a set V of size 3. The coordinate axes of $\mathbb{R}^V$ are indicated by black lines. The solid dot indicates the origin in $\mathbb{R}^V$ and the open dots are the standard basis vectors, which we can view as elements of V.

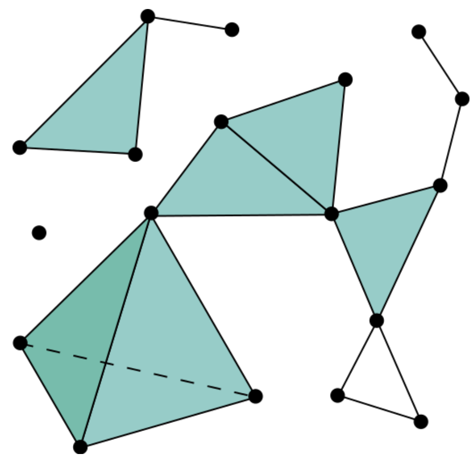

FIGURE 2. A simplicial complex, drawn as though embedded in $\mathbb{R}^3$. The 0-simplices are indicated by black dots, the 1-simplices are indicated by line segments and the color-filled regions indicate the presence of a higher dimensional simplices. There are eighteen 0-simplices, twenty-three 1-simplices, eight 2-simplices and one 3-simplex. The simplicial complex is disconnected with 3 components.

We can view $v \in V$ as the formal sum whose only term is v. In this way we can view $V \subset \mathbb{R}^V$ as the standard basis of $\mathbb{R}^V$. Moreover, since $\mathbb{R}^V$ is isomorphic to $\mathbb{R}^n$, with $n = |V|$, we can naturally place a topology on $\mathbb{R}^V$. The *simplex* σ_V on V is the smallest convex subset of $\mathbb{R}^V$ that contains V. See Figure 1. Concretely, we have

$$\sigma_V = \left\{ \sum_{v \in V} t_v v : t_v \geq 0, \text{ for all } v, \text{ and } \sum_{v \in V} t_v = 1 \right\}$$

from which we can also see that if $W \subset V$ then there is a natural inclusion $\iota_{VW} \colon \sigma_W \hookrightarrow \sigma_V$. The simplex σ_V is homeomorphic to the ball B^n of dimension n, where $n = |V| - 1$. We call σ_V an *n-simplex*.

Definition. An *abstract simplicial complex* is a collection Σ of nonempty sets satisfying the following condition: if $V \in \Sigma$ and W is a nonempty subset of S then $W \in \Sigma$. In symbols: $\emptyset \neq W \subset V \in \Sigma \Longrightarrow W \in \Sigma$.

Definition. Let Σ be an abstract simplicial complex. Let $\sim$ be the equivalence relation on $Y(\Sigma) = \bigsqcup_{V \in \Sigma} \sigma_V$ generated by $\big\{ \{x, \iota_{VW}(x)\} : \emptyset \neq W \subset V \in \Sigma \text{ and } x \in \sigma_W \big\}$. The *simplicial complex* $X(\Sigma)$ realizing Σ is the identification space $Y(\Sigma)/{\sim}$. See Figure 2.

It turns out that the quotient map $Y(\Sigma) \to X(\Sigma)$ restricts to an embedding on any simplex σ_V in $Y(\Sigma)$ and so we may view the σ_V with $V \in \Sigma$ as subspaces of $X(\Sigma)$. In fact, these subspaces are the closed cells (i.e. closures of cells) of a cellular complex structure on $X(\Sigma)$. Moreover, the intersection $\sigma_V \cap \sigma_{V'}$ of two simplices of X is precisely the simplex $\sigma_{V \cap V'}$ if $V \cap V' \neq \emptyset$ and is otherwise $\emptyset$.

(a) Let X be a simplicial complex with finitely many 0-simplices. Prove that X embeds as a subspace of $\mathbb{R}^n$, where n is the number of 0-simplices of X. Conclude that X is metrizable. (Hint: Let V be the set of 0-simplices of X. Can you find a natural way to embed X in the simplex $\sigma_V \subset \mathbb{R}^V$?)

(b) Prove that a simplicial complex is compact if and only if it has finitely many 0-simplices.[1] You may use the fact that a subset of a cellular complex is open if and only if its intersection with each closed cell is open.

[1] A general cellular complex is compact if and only if it has finitely many cells in total.