MATH 4530: HOMEWORK #10

- Due Friday, November 17th at 11pm.
- To be submitted on the course gradescope.
  - Please remember to indicate which pages your solutions to each problem appear on.
- §n: #m refers to exercise number m from the exercise list following Section n of the textbook (Topology, Second Edition by James R. Munkres).
- You are encouraged to discuss the problems in groups. *However, you must write your solutions individually!*

Problems.

(1) §32: #3
(2) §51: #2
(3) §51: #3
(4) A topological space X is *locally n-Euclidean* if every point $x \in X$ has a neighborhood U that is homeomorphic to $\mathbb{R}^n$. A *closed topological surface* is a nonempty, connected, locally 2-Euclidean, compact Hausdorff space. Prove that closed topological surfaces are metrizable. (Hint: Use §32: #3 and the Urysohn Metrization Theorem.)
(5) Let $X = X(\Sigma)$ be a compact simplicial complex. For each $k \in \mathbb{Z}$, let $\Delta_k = \Delta_k(X)$ denote the set

$$\Delta_k(X) = \{\sigma_V : V \in \Sigma \text{ with } \dim(\sigma_V) = k\}$$

of k-simplices of X. (Note that if k is negative then $\Delta_k = \emptyset$.) Let $C_k = C_k(X) = \mathbb{R}^{\Delta_k}$, that is $C_k(X)$ is the free real vector space on the set Δ_k. We view each k-simplex σ_V of X as a basis element of $C_k(X)$. Note that if $\Delta_k = \emptyset$ then C_k is the trivial real vector space $\mathbb{R}^0$, which we denote 0.

We fix an arbitrary total ordering on $\bigcup_{\sigma\in\Sigma}\sigma$. For $k \in \mathbb{Z}_+$, let $\partial_k\colon C_k \to C_{k-1}$ be the linear map that sends each k-simplex $\sigma_V \in C_k$ to the alternating sum $\sum_{i=0}^{k}(-1)^i\sigma_{V\setminus\{v_i\}} \in \mathcal{C}_{k-1}$ where v_i is the ith vertex in the restriction to V of the total ordering on $\bigcup_{\sigma\in\Sigma}\sigma$ (i.e. $V = \{v_0, v_1, v_2, \ldots, v_k\}$ with $v_0 < v_1 < v_2 < \cdots < v_k$). For nonpositive k, let ∂_k be the zero map.

Notice that, for any $k \in \mathbb{Z}$, the image $B_k = \mathrm{im}(\partial_{k+1})$ of ∂_{k+1} is contained in the kernel $Z_k = \ker(\partial_k)$ of ∂_k, or equivalently: the composition $\partial_{k+1} \circ \partial_k$ is the zero map. (You do not need to prove this in your solution.) The elements of the vector subspace $Z_k \subset C_k$ are called *k-cycles*. The elements of the vector subspace $B_k \subset C_k$ are called *k-boundaries*. We can think of k-cycles as "possible holes of dimension k" in X and of k-boundaries as k-cycles that are in fact "filled in" in X.

The sequence of maps

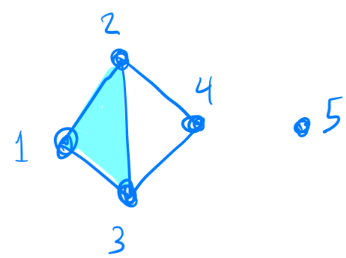
$$0 \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} C_3 \leftarrow \cdots$$

is called a *chain complex*.

The kth *homology group* of X in real coefficients is the quotient vector space $H_k(X) = Z_k/B_k$. Since we have quotiented out the "filled in" holes We can think of the elements of $H_k(X)$ as representing "actual holes" of dimension k in X. It turns out that the homology groups of X depend only on its topology (in fact only on its "homotopy type") and not on its simplicial structure: every homeomorphic (or "homotopy equivalent") simplicial complex has the same homology groups.

(a) (i) Compute the homology groups of $X = X(\Sigma)$ in real coefficients, where Σ is as follows.

$$\begin{aligned}\Sigma = \{\ &\{1\},\{2\},\{3\},\{4\},\{5\},\\ &\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\\ &\{1,2,3\}\ \}\end{aligned}$$



(ii) Find a basis for each nonzero homology group. In a couple of sentences, try to give an intuitive topological interpretation of how these basis elements relate to "holes" in X.

(b) Compute the homology groups of X in real coefficients, where X is the boundary $\partial\sigma_{\{1,2,3,4\}}$ of the 3-simplex $\sigma_{\{1,2,3,4\}}$, that is, $X = X(\Sigma)$ where Σ is as follows.

$$\begin{aligned}\Sigma = \{\ &\{1\},\{2\},\{3\},\{4\},\\ &\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{1,4\},\\ &\{1,2,3\},\{2,3,4\},\{1,2,4\},\{1,3,4\}\ \}\end{aligned}$$